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Dispersion of Solute in Laminar Flow through a Circular Tube

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Abstract

An exact closed-form solution of a mathematical model describing the transport of soluble matter in a solvent inside a circular tube has been obtained in terms of a confluent hypergeometric function and shown to be in excellent agreement with previously published experimental and numerical works.

INTRODUCTION

Dispersion of soluble matter in a solvent flowing through a tube is of practical interest. Several investigators (*1–16*) studied this problem both experimentally and theoretically in order to determine the concentration profile and mass-transfer rate in a fully developed laminar flow inside a circular tube.

GOVERNING EQUATION AND MATHEMATICAL ANALYSIS

The partial differential equation governing solute transport in a circular tube carrying a dilute solution was given originally by Taylor (*1*):

$$D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{\partial^2 c}{\partial z^2}\right] = V_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial c}{\partial z} + \frac{\partial c}{\partial t} \tag{1}$$

where D = molecular diffusivity; $V_{z,max}$ = maximum, center-line axial velocity; R = pipe radius; c = solute concentration; r and z = cylindrical coordinates; and t = time. The left-hand side of Eq. (1) describes spreading by molecular diffusion, and the right-hand side describes spreading by convection.

Assuming steady flow, a constant diffusion coefficient, and negligible axial diffusion, Eq. (1) in dimensionless form becomes

$$(1 - \rho^2)\frac{\partial C}{\partial \eta} = \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial C}{\partial \rho}\right)\right] \tag{2}$$

where $\rho = r/R$, $\eta = (z/R)/\mathrm{Pe}$, $C = c/C_0$.

The boundary conditions needed to solve Eq. (2) are

$$\text{B.C.1:} \quad \partial C/\partial \rho = 0 \text{ at } \rho = 1 \tag{3}$$

$$\text{B.C.2:} \quad C = 1 \text{ at } \eta = 0 \tag{4}$$

$$\text{B.C.3:} \quad C = 0 \text{ at } \rho = 1 \tag{5}$$

CLOSED-FORM SOLUTION

Equation (2) can be solved by the separation of variables technique by assuming

$$C(\rho,\eta) = R(\rho)Z(\eta)$$

which upon substitution into Eq. (2) is separated into the two following ordinary differential equations:

$$\frac{dZ}{d\eta} = -\beta^2 Z \tag{6}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \beta^2(1 - \rho^2)R = 0 \tag{7}$$

The solution of Eq. (6) is given by

$$Z = A_1 e^{-\beta^2 \eta} \tag{8}$$

where A_1 is any arbitrary constant.

Equation (7) is a Sturm-Liouiville system with β_n = real-numbered eigenvalues and $R_n(\rho)$ = eigenfunctions that form an orthogonal set with respect to the weight function $\rho(1 - \rho^2)$ over the interval [0,1].

Introducing the transformation $t = -\beta\rho^2$ into Eq. (7) and taking the Laplace transformation of the resulting equation, we get

$$\bar{R}(S) = \frac{A_2(S + 1/2)}{(S - 1/2)^{(\beta/4 + 1/2)}} \tag{9}$$

From the table of Laplace transforms (*17*), the Laplace inverse of Eq. (9) can be written as

where $M_{(\beta/4,0)}$ is the Wittaker function which can be generally written as

$$M_{\mu,\nu}(X) = X^{1/2+\nu} e^{-X/2} {}_1F_1[\tfrac{1}{2} + \omega - \mu;\ 2\nu + 1;\ X] \tag{11}$$

${}_1F_1$ = Kummer's confluent hypergeometric function which is defined by

$${}_1F_1\ [a;\ b;\ x] = \sum_{n=0}^{\infty} \frac{(a)_n X^n}{(b)_n n!} \tag{12}$$

By the substitution of Eq. (11) into Eq. (10) and rearrangement, we get the solution of Eq. (7) as

$$R(\rho) = A_2 e^{-\beta\rho^2/2} {}_1F_1\left[\frac{2 - \beta}{4};\ 1;\ \beta\rho^2\right] \tag{13}$$

The eigenvalues β_n of Eq. (13) can be evaluated from

$${}_1F_1\left[\frac{2 - \beta_n}{4};\ 1;\ \beta_n\right] = 0, \qquad \text{for } n = 1, 2, 3, \ldots \tag{14}$$

The complete solution of Eq. (2) which describes the solute concentration profile is

$$C(\rho,\eta) = \sum_{n=0}^{\infty} A_n e^{-\beta_n^2\eta} e^{-\beta_n\rho^2/2} {}_1F_1\left[\frac{2-\beta_n}{4}; 1; \beta_n\rho^2\right] \quad (15)$$

where A_n are arbitrary constants which can be numerically computed from

$$A_n = \frac{\int_0^1 \rho(1-\rho^2)R_n d\rho}{\int_0^1 \rho(1-\rho^2)R_n^2 d\rho} = \frac{-2}{\beta_n\left(\frac{\partial R_n}{\partial \beta_n}\right)_{\rho=1}} \quad (16)$$

RESULTS AND DISCUSSION

The dimensionless concentration C ($= c/C_0$) is plotted versus the dimensionless length in Fig. 1 using Eq. (15). Theoretical predictions of Eq. (15) were compared with previously published works (*18, 19*), and excellent agreement was obtained between our analytical solution and Koyama et al.'s (*18*) experimental data. The discrepancies between our exact solution and Newman's (*19*) numerical work are expected because his solution was an approximate solution.

NOMENCLATURE

A_1, A_2	constants in Eqs. (8) and (9), respectively
a, b	parameters in Eq. (12)
A_n	series coefficient in Eq. (15)
c	solute concentration
C	solute dimensionless concentration
C_0	solute inlet concentration
D	solute molecular diffusivity
${}_1F_1(a; b; x)$	confluent hypergeometric function
L	length of tube

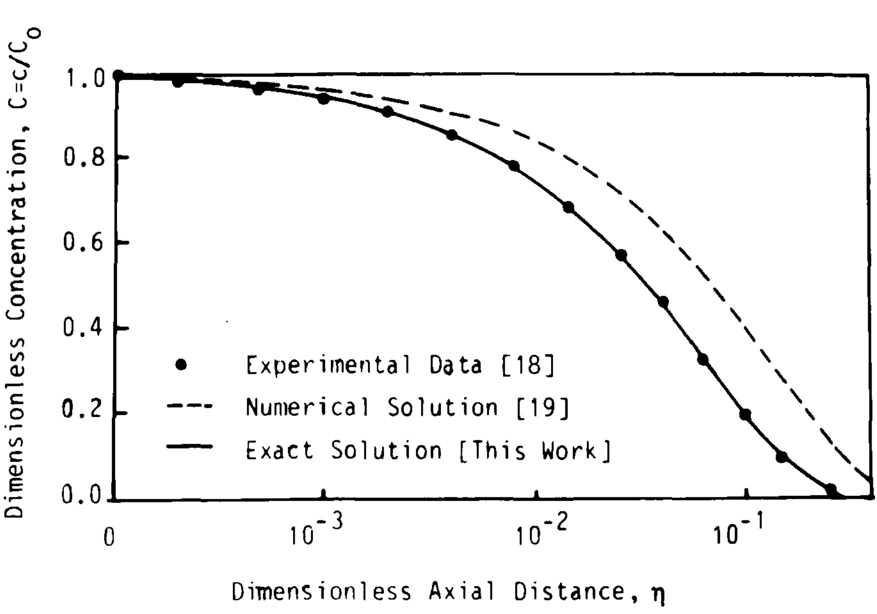

FIG. 1. Comparison of the solute distribution of the present solution with previously published experimental and numerical works.

$M_{\mu,\upsilon}(x)$	Wittaker function
Pe	mass transfer Peclet number
r	radial coordinate
R	tube radius
$\bar{R}(S)$	Laplace radial function of R (ρ)
s	Laplace parameter
$V_{z,max}$	fluid axial maximum velocity
z	axial coordinate
Z	function of z

Greek Symbols

β	eigenvalue of Eq. (13)
η	dimensionless axial coordinate
ρ	dimensionless radial coordinate

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